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Identification of a space-dependent source term in a nonlocal problem for the general time-fractional diffusion equation

Emilia Bazhlekova *, Ivan Bazhlekov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 8, Sofia 1113, Bulgaria

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ABSTRACT

The diffusion equation with a general convolutional derivative in time is considered on a bounded domain, as one of the boundary conditions is nonlocal. We are concerned with the inverse source problem of recovery of a space-dependent source term from given final time data. To find the source term and the solution, we resort to generalized eigenfunction expansion, using a bi-orthogonal pair of bases. Estimates for the time-dependent components in the spectral expansions are established and applied to prove uniqueness and existence in the classical sense. Analytical and numerical examples are provided.

1. Introduction

Anomalous diffusion processes are modeled by employing different types of fractional partial differential and integro-differential equations (see [1] and the references cited there). In particular, the power-law dependence on time of the mean squared displacement can be captured by time-fractional diffusion equations. However, most of the anomalous diffusion phenomena in complex systems do not show a mono-scaling behavior. Instead, transitions between different diffusion regimes in course of time are observed. One way to capture such multi-scaling behavior is by replacing the relatively simple operators of fractional derivatives by more general operators with specific memory kernels [2–4].

A large number of different generalizations of the classical fractional calculus operators have been proposed and extensively discussed recently, see e.g. [5,6] for some general unifying models of fractional calculus.

In this work we adopt the definition of generalized fractional derivative of Caputo type introduced in [7] (see also [8]) in the form

$$(\mathbb{D}_t^{(k)}f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(t-\tau)f(\tau)\,\mathrm{d}\tau - k(t)f(0), \quad t>0, \tag{1}$$

where $k(t)$ is a nonnegative locally integrable kernel. The exact assumptions imposed on the kernel will be specified later. Here we list some basic particular cases.

The Caputo fractional derivative of order $\alpha \in (0,1)$ is recovered from (1) for the power-law memory kernel $k(t) = \omega_{1-\alpha}(t)$, where

$$\omega_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad \beta>0, \quad t>0, \tag{2}$$

* Corresponding author.
E-mail addresses: e.bazhlekova@math.bas.bg (E. Bazhlekova), i.bazhlekov@math.bas.bg (I. Bazhlekov).

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with $\Gamma(\cdot)$ being the Gamma function. Other basic examples of memory kernels are the multi-term power-law memory kernel

$$k(t) = \sum_{j=1}^{m} q_j \omega_{1-\alpha_j}(t), \quad 0 < \alpha_j < 1, \quad q_j > 0, \quad j = 1, \dots, m,$$

the distributed-order memory kernel

$$k(t) = \int_0^1 \omega_{1-\alpha}(t) p(\alpha)\, \mathrm{d}\alpha,$$

where $p(\cdot)$ is a nonnegative weight function, and the truncated power-law memory kernel

$$k(t) = e^{-\gamma t}\omega_{1-\alpha}(t), \quad \gamma > 0, \quad 0 < \alpha < 1.$$

The time-fractional diffusion equation with the general integro-differential operator (1) is discussed in [2,3], where its relevance for describing a broad class of anomalous nonscaling patterns is pointed out. The Cauchy problem for the general diffusion equation on an unbounded space domain is studied in detail in [7]. In [9] some uniqueness and existence results, as well as a maximum principle, are established for the initial–boundary-value problem. Optimal estimates for the decay in time of solutions to the general time-fractional diffusion equations on a bounded domain subject to homogeneous Dirichlet boundary condition are deduced in [10], where it is shown that the different kernels can have very different kinds of decay, e.g. exponential, algebraic, or logarithmic. An initial value problem for a semilinear differential equation with the general fractional derivative (1) is studied in [11], where by using the Schauder fixed point theorem, the uniqueness and the local/global existence of solution are established.

Inverse problems for various types of fractional evolution equations are extensively studied recently, see e.g. [12–14]. Different kinds of inverse problems for the diffusion equations with the Caputo time-derivative are considered in [15–20]. For a comprehensive tutorial on inverse problems for anomalous diffusion processes we refer to [21]. Inverse problems for subdiffusion equations with a more general time-derivative, such as multi-term and distributed-order time-fractional equations, are also considered. Uniqueness for two kinds of inverse problems of identifying the orders of fractional derivatives in multi-term time-fractional diffusion equation is established in [22]. Uniqueness results for the recovering of weight function in distributed-order diffusion equations from one interior point observation of the solution are obtained in [23,24]. An inverse problem for the general fractional derivative is studied in [25] and the results are applied to determine time- and space-dependent sources in general time-fractional diffusion and wave equations. Existence, uniqueness and stability for the inverse source problem with final overdetermination for a generalized subdiffusion equation are established in [26].

In this work, we are concerned with the problem of determining a space-dependent source $h(x)$ and the solution $u(x,t)$ to the following nonlocal boundary-value problem with final overdetermination

$$\mathbb{D}_t^{(k)} u(x,t) = u_{xx}(x,t) + h(x), \quad x \in (0,1), \quad t \in (0,T), \tag{3}$$

$$u(1,t) = 0, \quad u_x(0,t) = u_x(1,t), \quad t \in (0,T], \tag{4}$$

$$u(x,0) = 0, \quad u(x,T) = g(x), \quad x \in [0,1], \tag{5}$$

where the operator $\mathbb{D}_t^{(k)}$ acting with respect to the time variable is defined in (1), $g(x)$ is a known square integrable function and $T > 0$ is the final time.

Due to the nonlocal character of the second boundary condition in (4), the corresponding spatial differential operator is non-selfadjoint and standard eigenfunction expansion technique does not apply. Non-selfadjoint operators appear e.g. in the modeling of processes with dissipation [27]. In many cases a nonlocal condition is more realistic in treating physical problems than the classical local conditions, which motivates the study of nonlocal boundary-value problems. Direct problems for diffusion equations with nonlocal boundary conditions are considered e.g. in [28–31]. Inverse source problems with nonlocal boundary conditions are studied e.g. in [32–36]. The papers [32] and [33] are concerned with particular cases of the inverse source problem (3)–(5) with $\mathbb{D}_t^{(k)} = \frac{\mathrm{d}}{\mathrm{d}t}$ and $\mathbb{D}_t^{(k)} = \mathbb{D}_t^{\alpha}$, the Caputo time-fractional derivative of order $\alpha \in (0,1)$, respectively. Inverse source problems for diffusion equations with other types of time-fractional operators and boundary conditions (4) are studied in [34,35].

The aim of this work is to construct spectral expansions for the source function $h(x)$ and solution $u(x,t)$ of problem (3)–(5) and to prove that these expansions provide a unique solution in the classical sense, based on estimates for the impulse-response solution of the $\mathbb{D}_t^{(k)}$-relaxation equation. Explicit expressions for the time-dependent components in the spectral expansions are derived for the basic memory kernels, with the main emphasis on the multi-term and the truncated power-law memory kernels. To this end we propose a Prabhakar-type generalization of the multinomial Mittag-Leffler function, introduced in [37]. To illustrate the analytical findings, numerical results are presented.

The rest of this paper is organized as follows. Section 2 contains preliminaries on Bernstein functions and functions of Mittag-Leffler type. In Section 3 the assumptions on the memory kernel k in the definition of the general fractional derivative are formulated and basic examples are given. The impulse-response solution to the general fractional relaxation equation is studied in Section 4. Formal spectral expansions for the solution of problem (3)–(5) and the source term are obtained in Section 5. In Section 6 estimates for the time-dependent components in the spectral expansions are established and used to prove the main result concerning uniqueness and existence in the classical sense. Explicit representations for the time-dependent components in the cases of the basic kernels are derived in Section 7. Section 8 contains analytical and numerical examples. Concluding remarks are given in Section 9.

2. Preliminaries

Most notations used throughout this paper are standard. The sets of positive integers, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{C}_+ = \{z \in \mathbb{C},\ \Re z > 0\}$.

By $\widehat{\varphi}(s)$ or $\mathcal{L}\{\varphi(t)\}(s)$ we denote the Laplace transform of the function $\varphi(t)$

$$\widehat{\varphi}(s) = \mathcal{L}\{\varphi(t)\}(s) = \int_0^\infty e^{-st}\varphi(t)\,\mathrm{d}t.$$

2.1. Bernstein functions and related classes of functions

We give definitions and some properties of four special classes of functions, related to Bernstein functions, which are necessary for the definition of the general fractional derivative. For a detailed exposition on these special classes of functions see [38].

A real-valued infinitely differentiable on $\mathbb{R}_+$ function $\varphi(t)$ is said to be a completely monotone function ($\mathcal{CMF}$) if

$$(-1)^n\varphi^{(n)}(t) \geq 0, \quad t > 0, \quad n \in \mathbb{N}_0.$$

A non-negative function φ defined on $\mathbb{R}_+$ is said to be a Bernstein function ($\varphi \in \mathcal{BF}$) if $\varphi'(t) \in \mathcal{CMF}$.

The class of Stieltjes functions ($\mathcal{SF}$) consists of all functions defined on $\mathbb{R}_+$ which have the representation (see [7])

$$\varphi(s) = \frac{a}{s} + b + \int_0^\infty e^{-s\tau}\psi(\tau)\,\mathrm{d}\tau, \quad s > 0, \tag{6}$$

where $a, b \geq 0$, $\psi \in \mathcal{CMF}$ and the Laplace transform of ψ exists for any $s > 0$.

A function φ defined on $\mathbb{R}_+$ is said to be a complete Bernstein functions ($\mathcal{CBF}$) if and only if

$$\varphi(s)/s \in \mathcal{SF}, \quad s > 0. \tag{7}$$

We have the inclusions: $\mathcal{SF} \subset \mathcal{CMF}$ and $\mathcal{CBF} \subset \mathcal{BF}$. Basic examples of Stieltjes and complete Bernstein functions are the following: $s^{-\alpha} \in \mathcal{SF}$ and $s^{\alpha} \in \mathcal{CBF}$ for $\alpha \in [0, 1]$.

A selection of properties is listed next.

Proposition 2.1. *The following properties are satisfied:*

(A) *$\varphi(s) \in \mathcal{SF}$, $\varphi \not\equiv 0$, if and only if $(s\varphi(s))^{-1} \in \mathcal{SF}$.*
(B) *$\varphi \in \mathcal{CBF}$, $\varphi \not\equiv 0$, if and only if $(\varphi + \lambda)^{-1} \in \mathcal{SF}$ for any $\lambda \geq 0$.*
(C) *If $\varphi \in \mathcal{SF}$ and $\psi \in \mathcal{CBF}$ then the composite function $\varphi(\psi) \in \mathcal{SF}$.*
(D) *Any function $\varphi \in \mathcal{BF}$ admits a continuous extension to $\overline{\mathbb{C}_+}$ which is analytic in $\mathbb{C}_+$.*
(E) *If $\varphi \in \mathcal{SF}$ or $\varphi \in \mathcal{CBF}$ then it can be analytically extended to $\mathbb{C}\backslash(-\infty, 0]$ and*

$$|\arg\varphi(z)| \leq |\arg z|, \ z \in \mathbb{C}\backslash(-\infty, 0].$$

Moreover, $\Im\{\varphi(z)\}\Im z \geq 0$ for $\varphi \in \mathcal{CBF}$ and $\Im\{\varphi(z)\}\Im z \leq 0$ for $\varphi \in \mathcal{SF}$.

For proofs of the statements in Proposition 2.1 we refer to [38], Theorems 6.2, 7.5, 7.9, and [39], Example 2.2 and Cor. 4.1.

2.2. Functions of Mittag-Leffler type

The three-parameter Mittag-Leffler function, also referred to as Prabhakar function, is defined as follows (see e.g. [40])

$$E^{\delta}_{\mu,\beta}(z) := \sum_{k=0}^{\infty} \frac{(\delta)_k}{k!}\frac{z^k}{\Gamma(\mu k + \beta)}, \qquad z \in \mathbb{C},\ \mu, \beta, \delta \in \mathbb{R},\ \mu > 0, \tag{8}$$

where $(\delta)_k$ denotes the Pochhammer symbol defined by

$$(\delta)_k = \frac{\Gamma(\delta + k)}{\Gamma(\delta)} = \delta(\delta + 1)\cdots(\delta + k - 1), \quad k \in \mathbb{N},\ \delta \in \mathbb{R}; \quad (\delta)_0 = 1,\ \delta \in \mathbb{R}\backslash\{0\}.$$

The classical Mittag-Leffler functions are obtained as particular cases of (8): $E_{\mu,\beta}(z) = E^1_{\mu,\beta}(z)$ and $E_{\mu}(z) = E_{\mu,1}(z) = E^1_{\mu,1}(z)$.

Recall the Laplace transform pair:

$$\mathcal{L}\left\{t^{\beta-1}E^{\delta}_{\mu,\beta}(at^{\mu})\right\}(s)=\frac{s^{\mu\delta-\beta}}{(s^{\mu}-a)^{\delta}},\quad \Re s>0,\ |s|>|a|^{\frac{1}{\mu}}. \tag{9}$$

The following relations can be established by the use of identity (9)

$$\int_0^t \tau^{\beta-1}E^{\delta}_{\mu,\beta}(a\tau^{\mu})\,\mathrm{d}\tau=t^{\beta}E^{\delta}_{\mu,\beta+1}(at^{\mu}), \tag{10}$$

$$\int_0^t (t-\tau)^{\beta-1}E^{\delta}_{\mu,\beta}(a(t-\tau)^{\mu})\tau^{\beta_0-1}E^{\delta_0}_{\mu,\beta_0}(a\tau^{\mu})\,\mathrm{d}\tau=t^{\beta+\beta_0-1}E^{\delta+\delta_0}_{\mu,\beta+\beta_0}(at^{\mu}). \tag{11}$$

The Mittag-Leffler function $E_{\mu,\beta}(-t)\in\mathcal{CMF}$ for $t>0$ if and only if $0<\mu\le 1$ and $\mu\le\beta$, see e.g. [41]. The complete monotonicity of the Prabhakar function is still under investigation [40].

For more details on the above functions of Mittag-Leffler type we refer to [40–44].

A multinomial generalization of the Prabhakar function (8), which will be used in this work, is proposed next.

Let us define the multinomial Prabhakar function as follows

$$E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(z_1,\ldots,z_m):=\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\frac{(\delta)_k}{k_1!\cdots k_m!}\frac{\prod_{j=1}^{m}z_j^{k_j}}{\Gamma\left(\beta+\sum_{j=1}^{m}\mu_jk_j\right)}, \tag{12}$$

where $z_j\in\mathbb{C}$, $\mu_j,\beta\in\mathbb{R}$, $\mu_j>0$, and k_j are nonnegative integers, $j=1,\ldots,m$.

In the particular case $\delta=1$ the Pochhammer symbol yields $(1)_k=k!$ and the function (12) is the multinomial Mittag-Leffler function defined in [37]

$$E_{(\mu_1,\ldots,\mu_m),\beta}(z_1,\ldots,z_m)=E^{1}_{(\mu_1,\ldots,\mu_m),\beta}(z_1,\ldots,z_m). \tag{13}$$

This function plays an essential role in the study of multi-term time-fractional diffusion equations. For some useful properties of the multinomial Mittag-Leffler function (13) in this context we refer to [45].

In the binomial case, $m=2$, the function (12) was recently introduced in [46], where it is written in the following equivalent form:

$$E^{\delta}_{(\mu_1,\mu_2),\beta}(z_1,z_2)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\delta)_{k_1+k_2}}{k_1!k_2!}\frac{z_1^{k_1}z_2^{k_2}}{\Gamma(\beta+\mu_1k_1+\mu_2k_2)}. \tag{14}$$

The standard Prabhakar function (8) is recovered from (12) for $m=1$.

Of particular importance for the study of multi-term time-fractional equations is the following function of a single variable t:

$$t^{\beta-1}E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(a_1t^{\mu_1},\ldots,a_mt^{\mu_m})=\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\frac{(\delta)_k}{k_1!\cdots k_m!}\frac{\left(\prod_{j=1}^{m}a_j^{k_j}\right)t^{\beta-1+\sum_{j=1}^{m}\mu_jk_j}}{\Gamma\left(\beta+\sum_{j=1}^{m}\mu_jk_j\right)},\quad t>0. \tag{15}$$

The Laplace transform of (15) is given as follows

$$\mathcal{L}\left\{t^{\beta-1}E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}\left(a_1t^{\mu_1},\ldots,a_mt^{\mu_m}\right)\right\}(s)=\frac{s^{-\beta}}{\left(1-\sum_{j=1}^{m}a_js^{-\mu_j}\right)^{\delta}}, \tag{16}$$

where $\Re s>0$, $\left|\sum_{j=1}^{m}a_js^{-\mu_j}\right|<1$. Identity (16) can be deduced by applying term-wise the Laplace transform to the series in (15) and taking into account the Laplace transform pair

$$\mathcal{L}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}(s)=s^{-\alpha},\qquad \Re s>0. \tag{17}$$

This yields

$$\mathcal{L}\left\{t^{\beta-1}E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}\left(a_1t^{\mu_1},\ldots,a_mt^{\mu_m}\right)\right\}(s)=s^{-\beta}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\frac{(\delta)_k}{k_1!\cdots k_m!}\prod_{j=1}^{m}\left(a_js^{-\mu_j}\right)^{k_j}.$$

To obtain (16) it remains to use the identity

$$\frac{1}{\left(1-\sum_{j=1}^{m}Z_j\right)^{\delta}}=\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\frac{(\delta)_k}{k_1!\cdots k_m!}\prod_{j=1}^{m}Z_j^{k_j}$$

with $Z_j=a_js^{-\mu_j}$, which follows from the binomial series

$$\frac{1}{(1-Z)^{\delta}}=\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}Z^k,\qquad |Z|<1,$$

and the multinomial theorem

$$(Z_1 + \cdots + Z_m)^k = \sum_{k_1+\cdots+k_m=k} \frac{k!}{k_1!\cdots k_m!} \prod_{j=1}^{m} Z_j^{k_j}. \tag{18}$$

The integration and convolution properties, (10) and (11), are extended to the multinomial case as follows

$$\int_0^t \tau^{\beta-1} E^{\delta}_{(\cdot),\beta}(\tau)\,\mathrm{d}\tau = t^{\beta} E^{\delta}_{(\cdot),\beta+1}(t), \tag{19}$$

$$\int_0^t (t-\tau)^{\beta-1} E^{\delta}_{(\cdot),\beta}(t-\tau)\tau^{\beta_0-1} E^{\delta_0}_{(\cdot),\beta_0}(\tau)\,\mathrm{d}\tau = t^{\beta+\beta_0-1} E^{\delta+\delta_0}_{(\cdot),\beta+\beta_0}(t), \tag{20}$$

where we have used the notation

$$E^{\delta}_{(\cdot),\beta}(t) := E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(a_1 t^{\mu_1}, \ldots, a_m t^{\mu_m}).$$

Properties (19) and (20) can be established by proving that Laplace transforms of both sides of the identities coincide, which follows from (16) and the convolution property of Laplace transform.

Let us note that for $\delta \in \mathbb{N}$ the Prabhakar function (13) can be represented as a multinomial Mittag-Leffler function (13) (with a larger m) by rewriting the denominator in the Laplace transform in (16) as a sum, using (18). For example, the following identity can be obtained in this way

$$E^2_{\mu,\beta}(at^{\mu}) = E_{(\mu,2\mu),\beta}(2at^{\mu}, -a^2 t^{2\mu}).$$

However, in such cases the notation as a Prabhakar function is more compact and convenient.

The multinomial Prabhakar function (12) is used in this work to find explicit expressions for the solution in the cases of multi-term and truncated power-law memory kernels. The necessity to consider functions with $\delta \neq 1$ is due to the nonlocal character of the boundary condition in (4).

3. General fractional derivative

First, let us specify the assumptions on the memory kernel $k(t)$ in the definition (1) of the general fractional derivative. In this work we assume that the Laplace transform $\widehat{k}(s)$ exists for all $s > 0$ and

$$\widehat{k}(s) \in \mathcal{SF}, \quad \lim_{s\to+\infty} s\widehat{k}(s) = +\infty, \tag{21}$$

where $\mathcal{SF}$ denotes the class of Stieltjes functions.

Let us note that the assumption $\widehat{k}(s) \in \mathcal{SF}$ implies the representation (6) for $\widehat{k}(s)$. Therefore, the kernel $k(t)$ is completely monotone for $t > 0$ and can have an additional term $b\delta(t)$, $b \geq 0$, where $\delta(\cdot)$ denotes the Dirac delta function. Moreover, $\widehat{k}(s) \in \mathcal{SF}$ is equivalent to $s\widehat{k}(s) \in \mathcal{CBF}$, see (7).

Concerning the limiting behavior of the kernel $k(t)$ the initial value theorem for the Laplace transform implies $k(0) = \lim_{s\to\infty} s\widehat{k}(s) = +\infty$. Thus, we restrict our attention to kernels, singular at the origin. In order to cover the basic physically meaningful examples of memory kernels, considered in [2–4], further restrictions on the limiting behavior of $\widehat{k}(s)$, respectively $k(t)$, are not required in this work.

Along with the kernel $k(t)$ we are also interested in the resolvent kernel $l(t) \in L^1_{loc}(\mathbb{R}_+)$, which is related to $k(t)$ by the following identity

$$(k * l)(t) \equiv 1, \tag{22}$$

where $*$ denotes the convolution

$$(\varphi * \psi)(t) = \int_0^t \varphi(t-\tau)\psi(\tau)\,\mathrm{d}\tau. \tag{23}$$

In Laplace domain (22) reads $\widehat{k}(s)\widehat{l}(s) = 1/s$. Therefore assumptions (21) imply $\widehat{l}(s) \in \mathcal{SF}$ (see property (A) in Proposition 2.1) and $\lim_{s\to\infty} \widehat{l}(s) = 0$. Hence $\widehat{l}(s)$ obeys representation (6) with $b = 0$. Therefore, under the assumptions (21) a resolvent kernel $l(t)$ exists and $l(t) \in \mathcal{CMF}$.

If together with assumptions (21) the kernel $k(t)$ satisfies also $\lim_{s\to\infty} \widehat{k}(s) = 0$ then the resolvent kernel $l(t)$ will satisfy (21) as well and, therefore, the kernels $k(t)$ and $l(t)$ can be switched. This is the case in any of the considered next basic examples of kernels, for more details see [2,4,10]. In what follows, by $\omega_{\beta}(t)$ we denote the power-law function defined in (2) and use the Laplace transform pair $\widehat{\omega}_{\beta}(s) = s^{-\beta}$.

Example 3.1. The power-law memory kernel

$$k(t) = \omega_{1-\alpha}(t), \quad \widehat{k}(s) = s^{\alpha-1}; \qquad l(t) = \omega_{\alpha}(t), \quad \widehat{l}(s) = s^{-\alpha}, \qquad 0 < \alpha < 1. \tag{24}$$

Example 3.2. The multi-term power-law memory kernel:

$$k(t)=\sum_{j=1}^{m} q_j\omega_{1-\alpha_j}(t),\quad \widehat{k}(s)=\sum_{j=1}^{m} q_j s^{\alpha_j-1}, \tag{25}$$

where $1>\alpha_1>\alpha_2>\cdots>\alpha_m>0$, $q_j>0$, $j=1,\ldots,m$, $m>1$. Without loss of generality we assume $q_1=1$. In this case

$$\widehat{l}(s)=\frac{1}{\sum_{j=1}^{m} q_j s^{\alpha_j}}=\frac{s^{-\alpha_1}}{1+\sum_{j=2}^{m} q_j s^{-(\alpha_1-\alpha_j)}}. \tag{26}$$

and the Laplace transform pair (16) yields the following representation for $l(t)$ as a multinomial Mittag-Leffler function

$$l(t)=t^{\alpha_1-1}E_{(\alpha_1-\alpha_2,\ldots,\alpha_1-\alpha_m),\alpha_1}\left(-q_2t^{\alpha_1-\alpha_2},\ldots,-q_mt^{\alpha_1-\alpha_m}\right). \tag{27}$$

In particular, in the two-term case ($m=2$)

$$k(t)=\omega_{1-\alpha_1}(t)+q\omega_{1-\alpha_2}(t),\;\; \widehat{k}(s)=s^{\alpha_1-1}+qs^{\alpha_2-1},\quad 1>\alpha_1>\alpha_2>0,\;\; q>0. \tag{28}$$

Therefore,

$$\widehat{l}(s)=s^{-\alpha_2}/(s^{\alpha_1-\alpha_2}+q),\;\; l(t)=t^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}(-qt^{\alpha_1-\alpha_2}), \tag{29}$$

where we have used the Laplace transform pair (9).

Example 3.3. The distributed-order memory kernel:

$$k(t)=\int_0^1 \omega_{1-\alpha}(t)p(\alpha)\,\mathrm{d}\alpha,\quad \widehat{k}(s)=\int_0^1 s^{\alpha-1}p(\alpha)\,\mathrm{d}\alpha, \tag{30}$$

where $p(\cdot)$ is a nonnegative weight function.

In the particular case of uniform distribution, $p\equiv 1$, the memory kernel becomes

$$k(t)=\int_0^1 \omega_{1-\alpha}(t)\,\mathrm{d}\alpha,\quad \widehat{k}(s)=\int_0^1 s^{\alpha-1}\,\mathrm{d}\alpha=\frac{s-1}{s\log s}. \tag{31}$$

Therefore

$$\widehat{l}(s)=\frac{\log s}{s-1}$$

and, by applying the Titchmarsh theorem for the inverse Laplace transform we get

$$l(t)=\int_0^\infty e^{-rt}K(r)\,dr,$$

where

$$K(r)=-\frac{1}{\pi}\Im\left\{\left.\frac{\log s}{s-1}\right|_{s=re^{i\pi}}\right\}=\frac{1}{r+1}.$$

This implies the representation

$$l(t)=\int_0^\infty \frac{e^{-rt}}{r+1}\,dr=e^t\mathcal{E}_1(t), \tag{32}$$

where $\mathcal{E}_1(t)$ denotes the exponential integral [47]

$$\mathcal{E}_1(t)=\int_t^\infty \frac{e^{-\xi}}{\xi}\,d\xi.$$

Any of the kernels $k(t)$ in the above examples can be considered in a weighted form, $e^{-\gamma t}k(t)$, where $\gamma>0$. Indeed, if the kernel $k(t)$ satisfies (21), then the Laplace transform relation

$$\mathcal{L}\{e^{-\gamma t}\varphi(t)\}(s)=\widehat{\varphi}(s+\gamma) \tag{33}$$

and property (C) in Proposition 2.1 imply that requirements (21) are satisfied for the kernel $e^{-\gamma t}k(t)$ as well. The next example is of this type.

Example 3.4. The truncated power-law memory kernel

$$k(t)=e^{-\gamma t}\omega_{1-\alpha}(t),\;\; \widehat{k}(s)=(s+\gamma)^{\alpha-1},\quad 0<\alpha<1,\;\; \gamma>0. \tag{34}$$

In this case $\widehat{l}(s) = (s+\gamma)^{1-\alpha}s^{-1}$ and, therefore, identities (9) and (33) imply the representation

$$l(t) = e^{-\gamma t} t^{\alpha-1} E_{1,\alpha}(\gamma t) . \tag{35}$$

Let us note that two other different representations of the kernel $l(t)$, resolvent to the truncated power-law memory kernel, are given in [10] and in [40].

In any of the above examples the kernels k and l can be switched, that is the kernel l can be taken as kernel k in the definition (1) for the operator $\mathbb{D}_t^{(k)}$. For instance, the operator (1) with the Mittag-Leffler memory kernel (29) emerges naturally in problems related to viscoelastic fluids, see e.g. [48], and in two-term fractional diffusion equations in modified form, see e.g. [2,49].

4. Inhomogeneous general fractional relaxation equation

Consider the inhomogeneous $\mathbb{D}_t^{(k)}$-relaxation equation

$$(\mathbb{D}_t^{(k)} v)(t) + \lambda v(t) = r(t), \quad \lambda \geq 0, \quad t > 0; \quad v(0) = 0. \tag{36}$$

Application of Laplace transform yields $s\widehat{k}(s)\widehat{v}(s) - \widehat{k}(s)v(0) + \lambda\widehat{v}(s) = \widehat{r}(s)$ and, thus $\widehat{v}(s) = \widehat{r}(s)/(s\widehat{k}(s)+\lambda)$. Taking inverse Laplace transform we obtain the solution of (36)

$$v(t) = \int_0^t G(t-\tau;\lambda) r(\tau)\, d\tau, \tag{37}$$

where $G(t;\lambda)$ is defined in Laplace domain as follows

$$\widehat{G}(s;\lambda) = \frac{1}{s\widehat{k}(s)+\lambda}, \quad \lambda \geq 0. \tag{38}$$

The function $G(t;\lambda)$ is the so-called impulse-response solution of Eq. (36). In the particular case $\lambda = 0$(38) yields $\widehat{G}(s;0)\widehat{k}(s) = 1/s$ and therefore

$$G(t;0) = l(t), \quad t \geq 0, \tag{39}$$

where $l(t)$ is the resolvent kernel of the kernel $k(t)$, see (22).

In the particular case of relaxation equation with the Caputo fractional derivative (Example 3.1) the impulse-response solution is expressed in terms of the Mittag-Leffler function $G(t;\lambda) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$. The relation $\int_0^T t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)\, dt = 1 - E_\alpha(-\lambda T^\alpha)$ implies the following property: there exists a constant $q > 0$, such that for any λ satisfying $\lambda \geq \lambda_1 > 0$ and any $T > 0$

$$0 < q \leq \lambda \int_0^T t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)\, dt < 1, \quad 0 < \alpha \leq 1. \tag{40}$$

Inequalities (40) turn out to be essential in the study of time-fractional diffusion equations, based on eigenfunction expansion. The bound from above is useful in the study of regularity of inhomogeneous equations, while the estimate from below is used in the study of inverse source problems with final overdetermination, see e.g. [15,18–20,33] for the proof of uniqueness or/and (conditional) stability results of such inverse problems.

It appears that estimates (40) can be extended to the case of a general memory kernel, satisfying (21). Next we summarize some properties of the impulse-response solution to the general equation (36), which will be used for the proof of uniqueness and existence of a classical solution to problem (3)–(5).

Lemma 4.1. *Under the assumptions (21) Eq. (36) has a unique solution given by (37), where the impulse-response solution $G(t;\lambda)$ obeys the following properties. For any $\lambda \geq 0$ the function $G(t;\lambda)$ is infinitely differentiable and $G(t;\lambda) \in \mathcal{CMF}$ with respect to $t \in \mathbb{R}_+$, $G(.;\lambda) \in L^1_{loc}(\mathbb{R}_+)$ and*

$$\int_0^T G(t;\lambda)\, dt > 0, \quad T > 0. \tag{41}$$

Moreover, for any fixed $T > 0$ and $\lambda_1 > 0$ there exists a constant $q = q(T,\lambda_1)$, such that for any $\lambda \geq \lambda_1 > 0$

$$0 < q \leq \lambda \int_0^T G(t;\lambda)\, dt < 1, \quad T > 0. \tag{42}$$

Proof. Since $s\widehat{k}(s) \in \mathcal{CBF}$, taking into account (38) and property (B) in Proposition 2.1 we deduce $\widehat{G}(s;\lambda) \in \mathcal{SF}$. Moreover, the limit in (21) implies

$$\lim_{s\to\infty} \widehat{G}(s;\lambda) = \lim_{s\to\infty} \frac{1}{s\widehat{k}(s)+\lambda} = 0$$

and, therefore, $b = 0$ in the representation (6) for the Stieltjes function $\widehat{G}(s;\lambda)$, i.e. it is a Laplace transform of a completely monotone function. Therefore, $G(t;\lambda) \in L^1_{loc}(\mathbb{R}_+)$ and $G(t;\lambda) \in \mathcal{CMF}$ for any fixed $\lambda \geq 0$.

Moreover, for any $\lambda \geq 0$

$$G(0;\lambda) = \lim_{s\to\infty} s\widehat{G}(s;\lambda) = \lim_{s\to\infty} \frac{s}{s\widehat{k}(s)+\lambda} = \lim_{s\to\infty} \frac{1}{\widehat{k}(s)} \neq 0,$$

since $1/\widehat{k}(s) \in \mathcal{CBF}$ and as such it is analytic nonnegative nondecreasing function (see properties (B) and (E) in Proposition 2.1). Therefore, $G(t;\lambda) > 0$ at least for $t \in (0,\varepsilon)$, which implies (41).

A proof of estimates (42) can be found in [50], Theorem 2.2. □

Let us note that for $\lambda > 0$ the impulse-response solution $G(t;\lambda)$ admits analytic extension to the right half-plane $\mathbb{C}_+$ and therefore it is a strictly positive function for all $t \in (0,\infty)$ (see e.g. [50], Theorem 2.2).

Two examples of explicit expressions for the impulse response solution are given next.

Example 4.2. Multi-term power-law memory kernel (25). Plugging (25) in (38) yields

$$\widehat{G}(s;\lambda) = \frac{1}{\sum_{j=1}^m q_j s^{\alpha_j} + \lambda} = \frac{s^{-\alpha_1}}{1 + \lambda s^{-\alpha_1} + \sum_{j=2}^m q_j s^{-(\alpha_1-\alpha_j)}}. \tag{43}$$

and by the use of (16) it follows

$$G(t;\lambda) = t^{\alpha_1-1} E_{(\alpha_1,\alpha_1-\alpha_2,\ldots,\alpha_1-\alpha_m),\alpha_1}\left(-\lambda t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}, \ldots, -q_m t^{\alpha_1-\alpha_m}\right). \tag{44}$$

Example 4.3. Truncated power-law memory kernel (34). Plugging (34) in (38) yields

$$\widehat{G}(s;\lambda) = \frac{1}{s(s+\gamma)^{\alpha-1} + \lambda} = \frac{(s+\gamma)^{-\alpha}}{1 + \lambda(s+\gamma)^{-\alpha} - \gamma(s+\gamma)^{-1}} \tag{45}$$

and by the use of identities (16) and (33) it follows

$$G(t;\lambda) = e^{-\gamma t} t^{\alpha-1} E_{(\alpha,1),\alpha}\left(-\lambda t^{\alpha}, \gamma t\right). \tag{46}$$

Representation (44) is well known, see e.g. [45], Theorem 2.2, where it is used to obtain regularity estimates for the multi-dimensional initial–boundary value problem for multi-term time-fractional diffusion equation. Since in Examples 4.2 and 4.3 the function $G(t;\lambda)$ is expressed in terms of multinomial Mittag-Leffler functions, part of the statements in Lemma 4.1 can be derived from the properties of these functions, established in [45].

5. Spectral expansions of the solution

The inner product in $L^2(0,1)$ is denoted by $\langle .,. \rangle$, i.e. $\langle f,g \rangle = \int_0^1 f(x)g(x)\,dx$. The norm $\|.\|$ in $L^2(0,1)$ is $\|f\| = \langle f,f \rangle^{1/2}$.

For the solution of the forward problem we apply a standard technique of spectral decomposition (see e.g. [28,29]). The eigenvalues of the spectral problem for the second order differential operator with the boundary conditions (4) are $4\pi^2 n^2$, $n \in \mathbb{N}_0$, as for $n \neq 0$ each eigenvalue has multiplicity 2. Therefore the system of eigenfunctions is not complete and must be supplemented with associated functions. In this way the following complete set of generalized eigenfunctions is obtained

$$X_{1,0}(x) = 2(1-x), \quad X_{1,n}(x) = 4(1-x)\cos\lambda_n x, \quad X_{2,n}(x) = 4\sin\lambda_n x, \quad \lambda_n = 2\pi n, \; n \in \mathbb{N}. \tag{47}$$

Any function $f \in L^2(0,1)$ admits a formal spectral expansion in the form

$$f(x) = f_{1,0}X_{1,0}(x) + \sum_{n=1}^{\infty}\left\{f_{1,n}X_{1,n}(x) + f_{2,n}X_{2,n}(x)\right\}. \tag{48}$$

This formal expansion has the uniqueness property: $f \equiv 0$ if and only if all coefficients $f_{j,n} = 0$. To determine the coefficients $f_{j,n}$ in (48) we use the bi-orthogonal set of functions

$$Y_{1,0}(x) = 1, \; Y_{1,n}(x) = \cos\lambda_n x, \; Y_{2,n}(x) = x\sin\lambda_n x, \quad \lambda_n = 2\pi n, \quad n \in \mathbb{N}, \tag{49}$$

satisfying $\langle X_{i,n}, Y_{j,m}\rangle = \delta_{ij}\delta_{nm}$. By taking inner product with $Y_{j,n}$ on both sides in (48) we deduce

$$f_{j,n} = \langle f, Y_{j,n}\rangle. \tag{50}$$

To solve the inverse source problem (3)–(5), we are looking for a source function $h(x)$ and a solution $u(x,t)$ in the form of the spectral expansions

$$h(x) = h_{1,0}X_{1,0}(x) + \sum_{n=1}^{\infty}\left\{h_{1,n}X_{1,n}(x) + h_{2,n}X_{2,n}(x)\right\}, \tag{51}$$

$$u(x,t) = u_{1,0}(t)X_{1,0}(x) + \sum_{n=1}^{\infty}\left\{u_{1,n}(t)X_{1,n}(x) + u_{2,n}(t)X_{2,n}(x)\right\}. \tag{52}$$

Inserting the above expansions in Eq. (3) and taking into account the initial condition $u(x,t) = 0$, we obtain the following system of equations

$$\mathbb{D}_t^{(k)}u_{1,n}(t) + \lambda_n^2 u_{1,n}(t) = h_{1,n}, \qquad u_{1,n}(0) = 0, \quad n \in \mathbb{N}_0, \tag{53}$$

$$\mathbb{D}_t^{(k)}u_{2,n}(t) + \lambda_n^2 u_{2,n}(t) = 2\lambda_n u_{1,n}(t) + h_{2,n}, \;\; u_{2,n}(0) = 0, \quad n \in \mathbb{N}. \tag{54}$$

Consecutive solution of Eqs. (53) and (54) by applying formula (37) yields

$$u_{1,n}(t) = h_{1,n}A_n(t), \qquad n \in \mathbb{N}_0, \tag{55}$$

$$u_{2,n}(t) = h_{2,n}A_n(t) + 2\lambda_n h_{1,n}B_n(t), \quad n \in \mathbb{N}. \tag{56}$$

Here the functions $A_n(t)$ and $B_n(t)$ are defined as follows

$$A_n(t) = \int_0^t G_n(\tau)\,d\tau, \qquad B_n(t) = \int_0^t\int_0^\tau G_n(\tau-\sigma)G_n(\sigma)\,d\sigma\,d\tau, \tag{57}$$

where $G_n(t) = G(t;\lambda_n^2)$ - the impulse-response solution (38) with $\lambda = \lambda_n^2$. From (38) and (57) the following representations in Laplace domain hold

$$\widehat{A}_n(s) = \frac{1}{s(s\widehat{k}(s)+\lambda_n^2)}, \qquad \widehat{B}_n(s) = \frac{1}{s(s\widehat{k}(s)+\lambda_n^2)^2}. \tag{58}$$

In this way we obtain the solution of the forward problem

$$u(x,t) = h_{1,0}A_0(t)X_{1,0}(x) + \sum_{n=1}^{\infty}\left\{h_{1,n}A_n(t)X_{1,n}(x) + (h_{2,n}A_n(t) + 2\lambda_n h_{1,n}B_n(t))X_{2,n}(x)\right\}. \tag{59}$$

Hence, the spectral expansion of the final time function $g(x) = u(x,T)$ is given by

$$g(x) = h_{1,0}A_0(T)X_{1,0}(x) + \sum_{n=1}^{\infty}\left\{h_{1,n}A_n(T)X_{1,n}(x) + (h_{2,n}A_n(T) + 2\lambda_n h_{1,n}B_n(T))X_{2,n}(x)\right\}. \tag{60}$$

By the uniqueness property of the spectral expansion (60) we get

$$g_{1,n} = h_{1,n}A_n(T), \;\; n \in \mathbb{N}_0; \qquad g_{2,n} = h_{2,n}A_n(T) + 2\lambda_n h_{1,n}B_n(T), \quad n \in \mathbb{N}. \tag{61}$$

We already know from (41) and (57) that $A_n(T) \neq 0$ for all $n \in \mathbb{N}_0$. Therefore, from (61) we arrive at the following expressions for the coefficients $h_{j,n}$ in the expansion (51), where $g_{j,n} = \langle g, Y_{j,n}\rangle$

$$h_{1,n} = \frac{1}{A_n(T)}g_{1,n}, \quad n \in \mathbb{N}_0; \qquad h_{2,n} = \frac{1}{A_n(T)}\left(g_{2,n} - 2\lambda_n g_{1,n}\frac{B_n(T)}{A_n(T)}\right), \quad n \in \mathbb{N}. \tag{62}$$

Plugging the coefficients $h_{j,n}$ in (59), we derive the coefficients $u_{j,n}(t)$ in the spectral expansion (52) of the solution $u(x,t)$

$$u_{1,n}(t) = g_{1,n}\frac{A_n(t)}{A_n(T)}, \quad n \in \mathbb{N}_0; \qquad u_{2,n}(t) = g_{2,n}\frac{A_n(t)}{A_n(T)} + 2\lambda_n g_{1,n}\left(\frac{B_n(t)}{A_n(T)} - \frac{A_n(t)B_n(T)}{A_n^2(T)}\right), \quad n \in \mathbb{N}. \tag{63}$$

In this way, plugging (62) in (51), and (63) in (52), we obtain the formal expansions for $h(x)$ and $u(x,t)$. The functions $A_n(t)$ and $B_n(t)$ in these expansions depend on the specific memory kernel.

Let us compare the obtained results with the results presented earlier in [32] and [33] for particular cases of the considered problem.

In [33] problem (3)–(5) is studied with $\mathbb{D}_t^{(k)} = \mathbb{D}_t^\alpha$ - the Caputo derivative of order $\alpha \in (0,1)$. In this case $s\widehat{k}(s) = s^\alpha$ and (9) and (58) yield the explicit representations

$$A_0(t) = \omega_{\alpha+1}(t), \quad A_n(t) = t^\alpha E_{\alpha,\alpha+1}(-\lambda_n^2 t^\alpha), \quad B_n(t) = t^{2\alpha}E^2_{\alpha,2\alpha+1}(-\lambda_n^2 t^\alpha), \quad n \in \mathbb{N}. \tag{64}$$

Problem (3)–(5) in the limiting case of the classical diffusion, $\mathbb{D}_t^{(k)} = \frac{\mathrm{d}}{\mathrm{d}t}$, is studied in [32], Problem 3. In this case we set $\alpha = 1$ in (64) and obtain

$$A_0(t) = t, \quad A_n(t) = \frac{1 - e^{-\lambda_n^2 t}}{\lambda_n^2}, \quad B_n(t) = \frac{1 - e^{-\lambda_n^2 t} - \lambda_n^2 t e^{-\lambda_n^2 t}}{\lambda_n^4}, \quad n \in \mathbb{N}.$$

Therefore, expansions (51)–(52) and (62)–(63) in this work are in agreement with the results obtained in [32]. However, our results for the case of Caputo derivative differ from those given in [33]. The reason is that the time-dependent terms $u_{2,n}(t)$ of the solution u given in [33], eq. (25), are incorrect, due to incorrect application of the Riemann–Liouville fractional derivative operator. This mistake, in turn, leads to a wrong representation for the source function in [33], eq. (30). Our representations (62) and (63), with time-dependent components given in (64), provide the correct solution in the case of Caputo time-fractional equation, studied in [33].

Let us note that the function $A_n(t)$ in (64) can be equivalently written as

$$A_n(t) = \frac{1 - E_\alpha(-\lambda_n^2 t^\alpha)}{\lambda_n^2},$$

which implies that $\lim_{n\to\infty} A_n(t) = 0$ (as $1/\lambda_n^2$). This property is important in the proof of convergence of the formal expansions and establishing regularity estimates for the time-fractional diffusion equation on a bounded domain with local boundary conditions. In the next section, bounds for the functions $A_n(t)$ and $B_n(t)$ in the case of general memory kernel are derived and used to prove that the obtained formal expansions define a solution in the classical sense.

6. Existence of a unique classical solution

In this section we prove that, under some assumptions on the final condition $g(x)$, the formal expansions (52)–(51) and (63)–(62) define a classical solution $u(x, t)$ and a continuous source function $h(x)$. For this reason we prove first some estimates for the time-dependent components $A_n(t)$ and $B_n(t)$.

Lemma 6.1. *The functions $A_0(t)$, $A_n(t)$, $B_n(t)$, $n \in \mathbb{N}$, are continuous on $[0,\infty)$, vanish at $t = 0$, positive and nondecreasing on $\mathbb{R}_+$. The following estimates for $t, T > 0$ and $n \in \mathbb{N}$ are satisfied:*

$$A_n(t) \le 1/\lambda_n^2, \tag{65}$$

$$\frac{1}{A_n(T)} \le C\lambda_n^2, \tag{66}$$

$$\frac{B_n(t)}{A_n(t)} \le 1/\lambda_n^2 \tag{67}$$

$$B_n(t) \le 1/\lambda_n^4, \tag{68}$$

where the constant $C = C(T) > 0$ is independent of n.

Proof. Taking into account (58) and the limiting behavior (21) of the kernel k we obtain by applying the initial value theorem for the Laplace transform

$$A_n(0) = \lim_{s\to\infty} s\widehat{A}_n(s) = \lim_{s\to\infty} \frac{1}{s\widehat{k}(s) + \lambda_n^2} = 0, B_n(0) = \lim_{s\to\infty} s\widehat{B}_n(s) = \lim_{s\to\infty} \frac{1}{(s\widehat{k}(s) + \lambda_n^2)^2} = 0.$$

Lemma 4.1, Eq. (41), implies that $A_0(t)$, $A_n(t)$, $B_n(t)$, $n \in \mathbb{N}$, are continuous on $[0,\infty)$, positive and nondecreasing.

Let now $n \in \mathbb{N}$. Estimates (65) and (66) are already proven in Lemma 4.1, see (42). Further, from the definitions (57), we obtain by applying (65) (here $*$ denotes the convolution (23))

$$B_n(t) = (1 * G_n * G_n)(t) = (A_n * G_n)(t) \le \frac{1}{\lambda_n^2}(1 * G_n)(t) = \frac{1}{\lambda_n^2} A_n(t),$$

which yields (67). This together with (65) implies (68). □

Along with (65)–(68), we distinguish two additional inequalities established in Lemma 6.1, which are necessary for the proof of uniqueness and convergence of the spectral expansions

$$A_n(T) > 0, \quad A_n(t) \le A_n(T), \quad 0 \le t < T, \quad n \in \mathbb{N}_0. \tag{69}$$

In fact, the second inequality in (69) is also strict. Indeed, since $G_n(t) \in \mathcal{CMF}$ (see Lemma 4.1) then $A_n(t) \in \mathcal{BF}$ and, as such, it admits a continuous extension to $\overline{\mathbb{C}}_+$, which is analytic in $\mathbb{C}_+$ (see property (D) in Proposition 2.1). Therefore, it is strictly positive as well as strictly increasing on $\mathbb{R}_+$.

The next theorem contains our main result.

Theorem 6.2. *Assume*

$$g \in C^3[0,1], \quad g(1) = g''(1) = 0, \quad g'(0) = g'(1). \tag{70}$$

Then there exists a unique solution (u, h) to problem (3)–(5), defined by the spectral expansions (51) and (52) with terms defined in (62) and (63). The function $u(x,t)$ is a classical solution, i.e. $u(.,t) \in C^2[0,1]$, $\mathbb{D}_t^{(k)} u(x,.) \in C(0,T]$ and the source function $h \in C[0,1]$.

Proof. To prove uniqueness we use the uniqueness property of the spectral expansions. Indeed, if $g \equiv 0$, then $g_{j,n} = 0$, and therefore, (62) and (63) imply $h_{j,n} = 0$, $u_{j,n} = 0$, i.e. all coefficients in the expansions (51) and (52) vanish, hence $h \equiv 0$ and $u \equiv 0$.

For the proof of the existence let us note first that under the conditions (70) on the function g repeated integration by parts yields for $n \in \mathbb{N}$

$$g_{1,n} = \langle g, Y_{1,n} \rangle = \frac{1}{\lambda_n^3} \langle g'''(x), \sin \lambda_n x \rangle, \tag{71}$$

$$g_{2,n} = \langle g, Y_{2,n} \rangle = -\frac{1}{\lambda_n^3} \langle x g'''(x), \cos \lambda_n x \rangle + \frac{3}{\lambda_n^4} \langle g'''(x), \sin \lambda_n x \rangle. \tag{72}$$

Then, applying (66), (67) and the inequalities $ab \le (a^2 + b^2)/2$ and $\lambda_n < \lambda_n^2$, identities (62) imply for $n \in \mathbb{N}$ and $x \in [0,1]$ (here and below the letter C denotes various positive constants)

$$\begin{aligned} |h_{1,n}| &\le \frac{|g_{1,n}|}{A_n(T)} \le C\lambda_n^2 |g_{1,n}| \\ &= \frac{C}{\lambda_n} \left| \langle g'''(x), \sin \lambda_n x \rangle \right| \\ &\le C \left(\frac{1}{\lambda_n^2} + \langle g'''(x), \sin \lambda_n x \rangle^2 \right), \end{aligned} \tag{73}$$

$$\begin{aligned} |h_{2,n}| &\le \frac{1}{A_n(T)} \left| g_{2,n} - 2\lambda_n g_{1,n} \frac{B_n(T)}{A_n(T)} \right| \le C\lambda_n^2 \left(|g_{1,n}| + |g_{2,n}| \right) \\ &\le \frac{C}{\lambda_n} \left(\left| \langle g'''(x), \sin \lambda_n x \rangle \right| + \left| \langle x g'''(x), \cos \lambda_n x \rangle \right| \right) \\ &\le C \left(\frac{1}{\lambda_n^2} + \langle g'''(x), \sin \lambda_n x \rangle^2 + \langle x g'''(x), \cos \lambda_n x \rangle^2 \right). \end{aligned} \tag{74}$$

The above estimates together with the uniform boundedness of $|X_{j,n}(x)|$ on $[0,1]$ yield by applying the Bessel inequality for the trigonometric series

$$|h_{1,0} X_{1,0}(x)| + \sum_{n=1}^{\infty} \left\{ |h_{1,n} X_{1,n}(x)| + |h_{2,n} X_{2,n}(x)| \right\} \le C \left(1 + \|g'''\|^2 \right) < \infty.$$

Therefore, the series expansion (51) for $h(x)$ is absolutely and uniformly convergent in $[0,1]$. Since all terms are continuous functions for $x \in [0,1]$, Weierstrass criterion yields $h \in C[0,1]$.

Further, estimates (66), (67), (68), and (69) imply

$$\left| \frac{A_n(t)}{A_n(T)} \right| \le 1, \quad \left| \frac{B_n(t)}{A_n(T)} - \frac{A_n(t) B_n(T)}{A_n^2(T)} \right| \le \left| \frac{B_n(t)}{A_n(T)} \right| + \left| \frac{B_n(T)}{A_n(T)} \right| \le \frac{C}{\lambda_n^2} \tag{75}$$

Identities (71), (72), estimates (75) and the fact that $g'''(x)$ is uniformly bounded on $[0,1]$ yield for the terms (63) of the expansion of $u(x,t)$

$$|u_{1,n}| \le |g_{1,n}| \le \frac{C}{\lambda_n^3}, \qquad |u_{2,n}| \le C(|g_{1,n}| + |g_{2,n}|) \le \frac{C}{\lambda_n^3}, \qquad n \in \mathbb{N},$$

which implies the absolute and uniform convergence for the series (52). This together with continuity of all terms for $x \in [0,1]$ and $t \in [0,T]$ implies continuity of the solution $u(x,t)$ on $[0,1] \times [0,T]$.

To show that $u(x,t)$ is a solution in the classical sense it remains to prove that the series obtained after term-wise differentiation $\sum_{j=1}^2 \sum_{n=j-1}^\infty u_{j,n}(t) X'_{j,n}(x)$, $\sum_{j=1}^2 \sum_{n=j-1}^\infty u_{j,n}(t) X''_{j,n}(x)$, and $\sum_{j=1}^2 \sum_{n=j-1}^\infty \mathbb{D}_t^{(k)} u_{j,n}(t) X_{j,n}(x)$ are absolutely and uniformly convergent.

Taking into account the identities for the basis functions (47)

$$X''_{1,n} = -\lambda_n^2 X_{1,n} + 2\lambda_n X_{2,n}, \qquad X''_{2,n} = -\lambda_n^2 X_{2,n},$$

estimates (75), $\lambda_n < \lambda_n^2$ for $n \ge 1$, we obtain for the series with the second derivatives in space

$$|u_{1,n} X''_{1,n}| \le C\lambda_n^2 |u_{1,n}| \le C\lambda_n^2 |g_{1,n}|, \quad |u_{2,n} X''_{2,n}| \le C\lambda_n^2 |u_{2,n}| \le C\lambda_n^2 (|g_{1,n}| + |g_{2,n}|).$$

Therefore, the situation is the same as for the series expansion of the function $h(x)$, see (73) and (74). By the use of (71) and (72) and the Bessel inequality it follows in an analogous way as above that the corresponding series are absolutely and uniformly convergent. For the other series the argument is similar.

The initial condition can be checked by letting $t \to 0$ under the summation sign and taking into account that $A_n(0) = 0$ and $B_n(0) = 0$. The two boundary conditions (4) hold by construction, since they are satisfied by the basis functions (47). □

Remark 6.3. By the linearity of the forward problem, the results obtained can be applied to solve problems with nonzero initial data $g_0 \not\equiv 0$. Indeed, the solution to a forward problem with a nonzero initial condition $u(x,0) = g_0(x)$, satisfying $g_0''(x) \in L^2(0,1)$ and the compatibility conditions $g_0(1) = 0$, $g_0'(0) = g_0'(1)$, can be represented as the sum: $g_0(x) + u(x,t)$, where u is the solution of the problem with zero initial condition (3)–(5), with source function $h(x) + g_0''(x)$ and over-determination condition $u(x,T) = g(x) - g_0(x)$.

7. Representations of $A_n(t)$ and $B_n(t)$

To use the spectral expansions (51) and (52) with terms defined in (62) and (63) we need representations of the time-dependent components $A_n(t)$ and $B_n(t)$. First, explicit expressions for the functions $A_n(t)$ and $B_n(t)$ are derived in the cases of the multi-term and the truncated power-law memory kernels.

Example 7.1 (*Multi-term power-law memory kernel* (25)). Plugging (44) into (57), and taking into account the integration and convolution properties for the multinomial Mittag-Leffler functions, (19) and (20), we obtain

$$A_n(t) = t^{\alpha_1} E_{(\alpha_1, \alpha_1-\alpha_2, \ldots, \alpha_1-\alpha_m), \alpha_1+1}\left(-\lambda_n^2 t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}, \ldots, -q_m t^{\alpha_1-\alpha_m}\right), \quad n \in \mathbb{N}_0, \tag{76}$$

$$B_n(t) = t^{2\alpha_1} E^2_{(\alpha_1, \alpha_1-\alpha_2, \ldots, \alpha_1-\alpha_m), 2\alpha_1+1}\left(-\lambda_n^2 t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}, \ldots, -q_m t^{\alpha_1-\alpha_m}\right), \quad n \in \mathbb{N}. \tag{77}$$

In particular,

$$A_0(t) = t^{\alpha_1} E_{(\alpha_1-\alpha_2, \ldots, \alpha_1-\alpha_m), \alpha_1+1}\left(-q_2 t^{\alpha_1-\alpha_2}, \ldots, -q_m t^{\alpha_1-\alpha_m}\right).$$

Example 7.2. Truncated power-law memory kernel (34). Plugging (46) into (57) yields

$$A_n(t) = \int_0^t G_n(\tau)\, \mathrm{d}\tau = e^{-\gamma t} \int_0^t e^{\gamma(t-\tau)} \tau^{\alpha-1} E_{(\alpha,1),\alpha}\left(-\lambda_n^2 \tau^\alpha, \gamma\tau\right) \mathrm{d}\tau. \tag{78}$$

In the particular case $n = 0$ this implies by the use of the convolution property (11)

$$A_0(t) = e^{-\gamma t} \int_0^t e^{\gamma(t-\tau)} \tau^{\alpha-1} E_{1,\alpha}(\gamma\tau)\, \mathrm{d}\tau = e^{-\gamma t} t^\alpha E^2_{1,\alpha+1}(\gamma t). \tag{79}$$

Similar compact representation cannot be obtained for the integral in (78) if $n \neq 0$. Instead, we can use the representation (14) of the binomial Mittag-Leffler function and take term-wise convolution with the function $e^{\gamma t}$. By the use of the identity

$$\int_0^t e^{\gamma(t-\tau)} \omega_\beta(\tau)\, \mathrm{d}\tau = t^\beta E_{1,1+\beta}(\gamma t), \tag{80}$$

where $\omega_\beta(\cdot)$ is given in (2), this yields the following series representation

$$A_n(t) = e^{-\gamma t} \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(k+l)!}{k!l!} (-\lambda_n^2)^k \gamma^l t^{\alpha k + l + \alpha} E_{1,\alpha k + l + \alpha + 1}(\gamma t). \tag{81}$$

Further, (46) and the convolution property (20) imply

$$(G_n * G_n)(t) = e^{-\gamma t} t^{2\alpha-1} E^2_{(\alpha,1),2\alpha}\left(-\lambda_n^2 t^\alpha, \gamma t\right).$$

Therefore, from (57)

$$B_n(t) = \int_0^t (G_n * G_n)(\tau)\, \mathrm{d}\tau = e^{-\gamma t} \int_0^t e^{\gamma(t-\tau)} \tau^{2\alpha-1} E^2_{(\alpha,1),2\alpha}\left(-\lambda_n^2 \tau^\alpha, \gamma\tau\right) \mathrm{d}\tau$$

and in an analogous way as above we obtain the series representation

$$B_n(t) = e^{-\gamma t} \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(k+l+1)!}{k!l!} (-\lambda_n^2)^k \gamma^l t^{\alpha k + l + 2\alpha} E_{1,\alpha k + l + 2\alpha + 1}(\gamma t). \tag{82}$$

For an alternative representation, via Laplace transform inversion, we give also the Laplace transform pairs, obtained from (34) and (58):

$$\widehat{A}_n(s) = \frac{z^{-(\alpha+1)}}{(1-\gamma z^{-1})\left(1+\lambda_n^2 z^{-\alpha}-\gamma z^{-1}\right)}, \quad \widehat{B}_n(s) = \frac{z^{-(2\alpha+1)}}{(1-\gamma z^{-1})\left(1+\lambda_n^2 z^{-\alpha}-\gamma z^{-1}\right)^2}, \quad z = s+\gamma. \tag{83}$$

By the use of (16) and (33) we derive from (83) the following representations in terms of multinomial Mittag-Leffler functions

$$A_n(t) = e^{-\gamma t} t^{\alpha} E_{(\mu_1,\dots,\mu_4),\alpha+1}\left(a_1 t^{\mu_1},\dots,a_4 t^{\mu_4}\right), \tag{84}$$

where $\mu_1 = \alpha$, $\mu_2 = 1$, $\mu_3 = \alpha+1$, $\mu_4 = 2$; $a_1 = -\lambda_n^2$, $a_2 = 2\gamma$, $a_3 = \gamma\lambda_n^2$, $a_4 = -\gamma^2$, and

$$B_n(t) = e^{-\gamma t} t^{2\alpha} E_{(\nu_1,\dots,\nu_8),2\alpha+1}\left(b_1 t^{\nu_1},\dots,b_8 t^{\nu_8}\right), \tag{85}$$

where $\nu_j = \mu_j$, $j = 1,\dots,4$, $\nu_5 = 2\alpha$, $\nu_6 = 2\alpha+1$, $\nu_7 = \alpha+2$, $\nu_8 = 3$; $b_1 = -2\lambda_n^2$, $b_2 = 3\gamma$, $b_3 = 4\gamma\lambda_n^2$, $b_4 = -3\gamma^2$, $b_5 = -\lambda_n^4$, $b_6 = \gamma\lambda_n^4$, $b_7 = -2\gamma^2\lambda_n^2$, $b_8 = \gamma^3$.

In order to use series (51) and (52) for numerical computation of the solution we need to evaluate numerically the time-dependent components $A_n(t)$ and $B_n(t)$. The simplest way to numerically evaluate the (multinomial) Mittag-Leffler function, its Prabhakar version, or other functions given by infinite series is to approximate the function by a truncated series. It is known, however, see e.g the discussion in [40], that in the case of Mittag-Leffler-type functions of negative argument this could lead to computational problems. Moreover, usually, there are no explicit representations available for $A_n(t)$ and $B_n(t)$ in terms of special functions. Such an example is the distributed-order memory kernel, even if the distribution is uniform. Therefore, alternative representations, appropriate for numerical computation, are needed.

In the next theorem, integral representations of the functions $A_n(t)$ and $B_n(t)$ are derived by inversion of the Laplace transforms in (58).

Theorem 7.3. *Assume the kernel $k(t)$ satisfies conditions (21), $\lim_{s\to 0} s\widehat{k}(s) = 0$ and $\Im\{\widehat{k}(re^{i\pi})\} \neq 0$. Then the following representations hold for $n \in \mathbb{N}$ and $t > 0$*

$$A_n(t) = \frac{1}{\lambda_n^2} - \int_0^\infty e^{-rt} K_n^A(r)\,dr, \tag{86}$$

and

$$B_n(t) = \frac{1}{\lambda_n^4} - \int_0^\infty e^{-rt} K_n^B(r)\,dr, \tag{87}$$

where $\lambda_n = 2\pi n$ and the functions $K_n^A(r)$ and $K_n^B(r)$ are defined as follows

$$K_n^A(r) = \frac{1}{\pi r}\frac{I(r)}{(R(r)+\lambda_n^2)^2+(I(r))^2}, \quad K_n^B(r) = \frac{2}{\pi r}\frac{(R(r)+\lambda_n^2)I(r)}{((R(r)+\lambda_n^2)^2+(I(r))^2)^2}, \tag{88}$$

with

$$R(r) = \Re\{s\widehat{k}(s)\}, \quad I(r) = \Im\{s\widehat{k}(s)\}, \quad s = re^{i\pi}.$$

Proof. Taking the inverse Laplace integral of $\widehat{A}_n(s)$ given in (58) we obtain

$$A_n(t) = \frac{1}{2\pi i}\int_{Br} \frac{e^{st}}{s(s\widehat{k}(s)+\lambda_n^2)}\,ds, \tag{89}$$

where $Br = \{s;\ \Re s = \sigma\}$, with $\sigma > 0$, is the Bromwich path. The function under the integral sign in (89) has no poles in the complex plane cut along the negative real axis, since $\Im\left\{s\widehat{k}(s)+\lambda_n^2\right\} \neq 0$ for $|\arg s| < \pi$ (due to the fact that $s\widehat{k}(s) \in \mathcal{CBF}$ and property (E) in Proposition 2.1). Let us bend the contour Br into the Hankel path $Ha(\rho)$, which starts from $-\infty$ along the lower side of the negative real axis, encircles the disc $|s| = \rho$ counterclockwise and ends at $-\infty$ along the upper side of the negative real axis. The integral on the circular contour $|s| = \rho$ equals $1/\lambda_n^2$ when $\rho \to 0$. This can be obtained by applying Jordan's lemma or by direct check and taking into account that

$$\lim_{s\to 0}\frac{s}{s(s\widehat{k}(s)+\lambda_n^2)} = \frac{1}{\lambda_n^2}.$$

The sum of the integrals along the lower and the upper sides of the negative real axis yields the real integral in (86) where

$$K_n^A(r) = -\frac{1}{\pi}\Im\left\{\left.\frac{1}{s(s\widehat{k}(s)+\lambda_n^2)}\right|_{s=re^{i\pi}}\right\}, \tag{90}$$

which implies the representation of $K_n^A(r)$ in (88). The proof for $B_n(t)$ is analogous. □

Let us note that $K_n^A(r) \geq 0$, which is in agreement with the fact that $A(t) \in \mathcal{BF}$, see (57). Indeed, the assumption $\widehat{k}(s) \in \mathcal{SF}$ implies $I(r) \geq 0$ by the use of property (E) in Proposition 2.1.

Theorem 7.3 is useful for the numerical computation of $A_n(t)$ and $B_n(t)$ in the cases of multi-term and distributed-order kernels. The functions $R(r)$ and $I(r)$ in (88) are defined in these cases as follows: in the multi-term case (25) implies

$$R(r) = \sum_{j=1}^{m} q_j r^{\alpha_j} \cos \alpha_j \pi, \quad I(r) = \sum_{j=1}^{m} q_j r^{\alpha_j} \sin \alpha_j \pi;$$

in the distributed-order case (34) implies

$$R(r) = \int_0^1 p(\alpha) r^{\alpha} \cos \alpha\pi \, \mathrm{d}\alpha, \quad I(r) = \int_0^1 p(\alpha) r^{\alpha} \sin \alpha\pi \, \mathrm{d}\alpha.$$

8. Analytical and numerical solutions

In this section we consider four examples of solutions to problem (3)–(5) with different choices of the kernel $k(t)$ and overdetermination function $g(x)$.

Example 8.1. Power-law kernel $k(t) = \omega_{1-\alpha}(t)$, $0 < \alpha < 1$, and overdetermination function $g(x) = 1 - x - a \sin 2\pi x$.

Since in the spectral expansion of the function $g(x)$ with respect to the basis $X_{j,n}$ only the following two coefficients do not vanish $g_{1,0} = 0.5$, $g_{2,1} = -a/4$, Eqs. (51), (52), (62), (63), yield

$$h(x) = \frac{1-x}{A_0(T)} - \frac{a \sin 2\pi x}{A_1(T)}, \quad u(x,t) = (1-x)\frac{A_0(t)}{A_0(T)} - a \sin 2\pi x \frac{A_1(t)}{A_1(T)}. \tag{91}$$

Plugging $A_0(t)$ and $A_1(t)$ from (64) yields

$$h(x) = (1-x)\frac{\Gamma(\alpha+1)}{T^{\alpha}} - \frac{a \sin 2\pi x}{T^{\alpha} E_{\alpha,\alpha+1}(-4\pi^2 T^{\alpha})}, \quad u(x,t) = (1-x)\frac{t^{\alpha}}{T^{\alpha}} - a \sin 2\pi x \frac{t^{\alpha} E_{\alpha,\alpha+1}(-4\pi^2 t^{\alpha})}{T^{\alpha} E_{\alpha,\alpha+1}(-4\pi^2 T^{\alpha})}.$$

For the numerical evaluation of the solution, computation of the Mittag-Leffler function is needed, which can be done by truncation of the defining series or applying Theorem 7.3.

Example 8.2. Truncated power-law kernel $k(t) = e^{-\gamma t}\omega_{1-\alpha}(t)$, $0 < \alpha < 1$, $\gamma > 0$, and function $g(x)$ defined as in Example 8.1.

The solution is given by (91), where $A_0(t)$ and $A_1(t)$ are defined in Example 7.2. For the numerical computation of the solution one can use the representation (79) for $A_0(t)$ and any of the representations (78), (81), or (84) for $A_1(t)$, as the infinite series are used after truncation.

Example 8.3. Distributed-order memory kernel with uniform distribution (31), $T = 1$, and $g(x) = (1-x)(2 + \cos 4\pi x)$.

For the chosen kernel

$$A_0(t) = \int_0^t e^{\tau} \mathcal{E}_1(\tau) \, d\tau,$$

where $\mathcal{E}_1(t)$ is the exponential integral, see Example 3.3. To evaluate numerically $A_n(t)$ and $B_n(t)$ for $n \in \mathbb{N}$ we use Theorem 7.3, where

$$R(r) = -\frac{(r+1)\log r}{\log^2 r + \pi^2}, \quad I(r) = \frac{\pi(r+1)}{\log^2 r + \pi^2}.$$

Numerical results for the functions $A_n(t)$ and $B_n(t)$ ($n = 1, 2, 3$) are presented in Fig. 1. Since the function $A_0(t)$ admits much larger values than $A_n(t)$, $n \in \mathbb{N}$, it is not plotted in the figure.

To find the solution, we notice that the non-vanishing coefficients in the spectral decomposition of the function $g(x)$ are $g_{1,0} = 1$ and $g_{1,2} = 0.25$. Therefore, (51), (52), (62), and (63) imply

$$h(x) = \frac{2(1-x)}{A_0(T)} + \frac{(1-x)\cos 4\pi x}{A_2(T)} - 8\pi \sin 4\pi x \frac{B_2(T)}{A_2^2(T)},$$

$$u(x,t) = 2(1-x)\frac{A_0(t)}{A_0(T)} + (1-x)\cos 4\pi x \frac{A_2(t)}{A_2(T)} + 8\pi \sin 4\pi x \left(\frac{B_2(t)}{A_2(T)} - \frac{A_2(t)B_2(T)}{A_2^2(T)}\right).$$

Numerical results for the source function $h(x)$ and the solution $u(x,t)$ are presented in Fig. 2.

Example 8.4. Assume $k(t)$ is the two-term power-law kernel (28) with $\alpha_1 = 0.8, \alpha_2 = 0.5$, $q = 1$. Let $T = 1$ and $g(x) = 3x^4 - 8x^3 + 6x^2 - 2x + 1$.

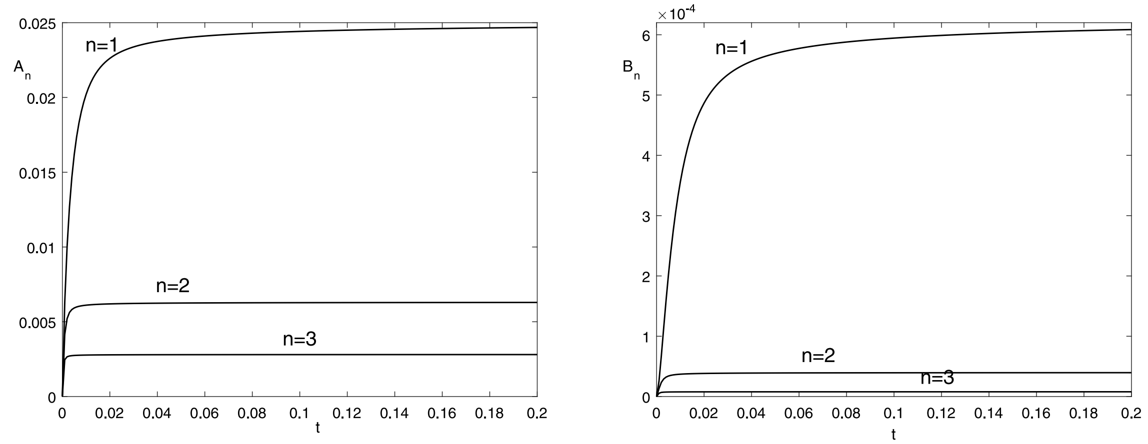

Fig. 1. Functions $A_n(t)$ (left) and $B_n(t)$ (right), $n = 1, 2, 3$, in the case of Example 8.3.

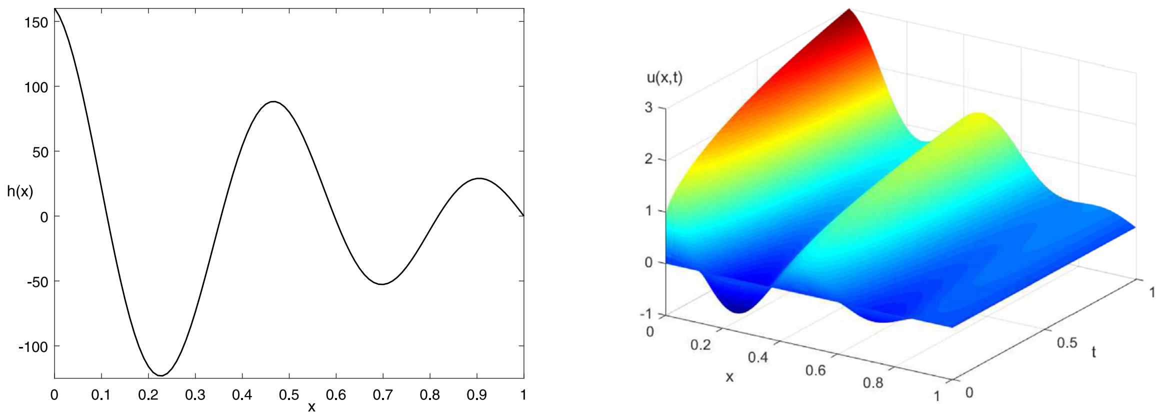

Fig. 2. Source function $h(x)$ (left) and solution $u(x,t)$ (right) for Example 8.3.

Applying (76) and (77) we obtain the following explicit expressions for the time-dependent components

$$A_0(t) = t^{\alpha_1} E_{\alpha_1-\alpha_2,\alpha_1+1}\left(-t^{\alpha_1-\alpha_2}\right),$$
$$A_n(t) = t^{\alpha_1} E_{(\alpha_1,\alpha_1-\alpha_2),\alpha_1+1}\left(-\lambda_n^2 t^{\alpha_1}, -t^{\alpha_1-\alpha_2}\right), \quad n \in \mathbb{N},$$
$$B_n(t) = t^{2\alpha_1} E^2_{(\alpha_1,\alpha_1-\alpha_2),2\alpha_1+1}\left(-\lambda_n^2 t^{\alpha_1}, -t^{\alpha_1-\alpha_2}\right), \quad n \in \mathbb{N}.$$

To evaluate numerically $A_n(t)$ and $B_n(t)$ for $n \in \mathbb{N}$ we have used the alternative representations (86) and (87) in Theorem 7.3, where

$$R(r) = r^{\alpha_1}\cos\alpha_1\pi + r^{\alpha_2}\cos\alpha_2\pi, \quad I(r) = r^{\alpha_1}\sin\alpha_1\pi + r^{\alpha_2}\sin\alpha_2\pi.$$

For the numerical computation of $A_0(t)$ we have used the formula

$$A_0(t) = \frac{1}{\pi}\int_0^\infty \frac{1-e^{-rt}}{r^{\alpha_2+1}} \cdot \frac{\sin\alpha_2\pi + r^{\alpha_1-\alpha_2}\sin\alpha_1\pi}{r^{2(\alpha_1-\alpha_2)} + 2r^{\alpha_1-\alpha_2}\cos(\alpha_1-\alpha_2)\pi + 1}\,\mathrm{d}r, \tag{92}$$

obtained by taking into account $A_0(t) = \int_0^t l(\tau)\,\mathrm{d}\tau$, (29), and the integral formula (A.32) in [41] for the Mittag-Leffler type function.

For illustration, plots of the functions $A_n(t)$, $n = 1, 2, 3, 4$, are given in Fig. 3. Since the function $A_0(t)$ admits much larger values than $A_n(t)$, $n \in \mathbb{N}$, it is not plotted in the figure.

The functions $h(t)$ and $u(x,t)$ are evaluated numerically by using expansions (51) and (52) by truncating the infinite series after 100 terms and the results are presented in Figs. 3 and 4. A comparison with 200 terms after truncation shows practically identical results. Therefore, taking 100 terms in the series is sufficient to reach a good accuracy. In fact, computation with 20 terms was also performed and the results differ only for very small x. A comparison with the exact solution $u(x,1) = g(x)$ is given in Fig. 4, which indicates a very good agreement between both solutions.

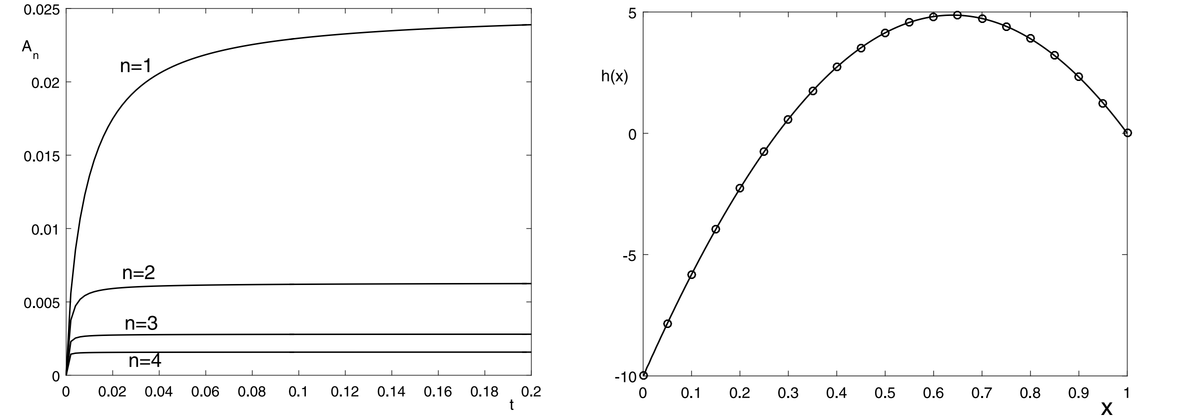

Fig. 3. Functions $A_n(t)$, $n = 1, 2, 3, 4$, (left) and source function $h(x)$ (right) for Example 8.4. The function $h(x)$ is computed by truncating the infinite series in (62) after 100 terms, compared to computation with 200 terms ($\circ$).

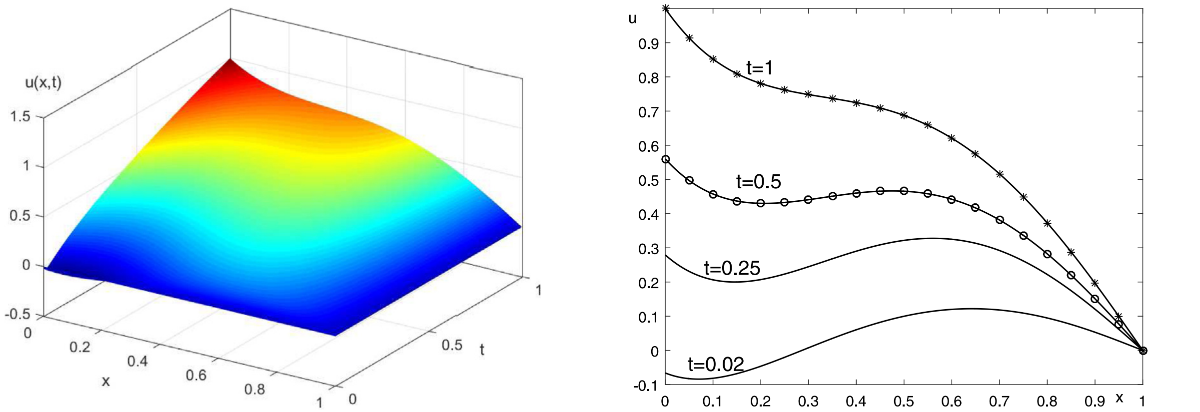

Fig. 4. The solution $u(x, t)$ for Example 8.4, evaluated by truncating the infinite series in (63) after 100 terms, compared for $t = 0.5$ to computation with 200 terms ($\circ$). The solution $u(x, t)$ for $t = 1$ is compared to the exact solution $g(x)$ ($*$).

In Examples 8.3 and 8.4, for the numerical computation of the improper integrals the MATLAB subroutine "integral" is used.

9. Concluding remarks

In this work we study an inverse source problem for the one-dimensional diffusion equation with a general convolutional derivative in time. Since one of the imposed boundary conditions is nonlocal, we resort to generalized eigenfunction expansion, using a bi-orthogonal pair of bases. Estimates for the time-dependent components in the spectral expansions are established and applied to prove uniqueness and existence in the classical sense. These estimates are deduced from the general assumptions imposed on the memory kernel by applying Laplace transform technique. They do not depend on the specific memory kernel and, therefore, allow a unified approach to the considered class of diffusion equations. This approach is mostly parallel to the one applied in the case of standard time-fractional diffusion equation.

Three basic particular cases of memory kernels are discussed in detail: the multi-term, the distributed order and the truncated power-law memory kernels. To find explicit expressions for the solutions in the cases of multi-term and truncated power-law memory kernels the multinomial Mittag-Leffler function, or its Prabhakar generalization, are used. The necessity to consider Prabhakar-type multinomial function is due to the nonlocal character of one of the boundary conditions. It is worth noting that in the basic particular cases, even when explicit representations are available, the properties of the time-dependent components are not evident. Therefore, as a by-product, we obtain some properties of the involved multinomial Mittag-Leffler functions.

The derived spectral expansions and estimates for the time-dependent components in the expansions can serve as a basis for establishing stability results for problem (3)–(5) in different settings. For example, in this way one can prove the following bound in Sobolev spaces for the recovered source function $h(x)$

$$\|h\|_{L^2(I)} \leq C\|g\|_{H^2(I)},$$

where $g(x) = u(x, T)$ is the overdetermination function and $I = (0, 1)$. Such an estimate is known for the inverse source problem for the classical diffusion case, as well as for the time-fractional diffusion equation with Caputo derivative and Dirichlet boundary conditions, see e.g. [21].

The performed numerical experiments are useful to get insight in the behavior of the plotted functions, to verify some of the analytical findings and to formulate open questions. For example, we observe that in the considered numerical examples the source function $h(x)$ and the solution $u(x, t)$ admit negative values, although the initial and final functions are chosen to be nonnegative ($u(x, 0) = 0$, $u(x, T) = g(x) \geq 0$), see Fig. 2, Fig. 3(right) and Fig. 4. This observation raises the question for further mathematical study and adequate physical interpretation of the model.

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